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High Order Explicit Hybrid Methods for Solving Second-Order Ordinary Differential Equations

(Kaedah Hibrid Tak Tersirat Peringkat Tinggi bagi Menyelesaikan Persamaan Pembezaan Biasa Peringkat-Dua)

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ABSTRACT

Two explicit hybrid methods with algebraic order seven for the numerical integration of second-order ordinary differential equations of the form $y'' = f(x, y)$ are developed. The algebraic order of these methods is the highest in comparison with other explicit hybrid methods of the same class. Numerical comparisons carried out show the advantage of the new methods.

Keywords: Algebraic order; explicit hybrid method; second-order ordinary differential equations

ABSTRAK

Dua kaedah hibrid tak tersirat dengan peringkat algebra tujuh untuk pengamiran berangka persamaan pembezaan biasa peringkat-dua berbentuk $y'' = f(x, y)$ dibangunkan. Peringkat algebra bagi kaedah-kaedah ini adalah yang tertinggi berbanding dengan kaedah hibrid tak tersirat lain dalam kelas yang sama. Perbandingan berangka yang dilakukan menunjukkan kelebihan bagi kaedah-kaedah baru ini.

Kata kunci: Kaedah hibrid tak tersirat; peringkat algebra; persamaan pembezaan biasa peringkat-dua

INTRODUCTION

There has been a great interest in the research of new methods for numerically solving the second-order ordinary differential equations of the form

$$y'' = f(x, y),\ y(x_0) = y_0,\ y'(x_0) = y'_0 \tag{1}$$

Such problems often arise in science and engineering field such as celestial mechanics, molecular dynamics, semi-discretizations of wave equations and so on. The second-order equation can be directly solved by using Runge-Kutta-Nystrom (RKN) methods or multistep methods. Several authors such as Hairer (1979), Cash (1981), Fatunla et. al.(1999) and Chawla (1984) have proposed hybrid methods which are obtained from the idea underlying both the Runge Kutta and linear multistep methods.

In the development of numerical methods for solving (1), it is important to pay attention at the algebraic order of the method because this is the main factor to achieve high accuracy. If the theoretical solution of the problem has a periodic nature, then it is also essential to consider the phase-lag and dissipation. These are actually two types of truncation errors. The first is the angle between the analytical solution and the numerical solution while the second is the distance from a standard cyclic solution. A pioneering work on phase-lag property has been done by Brusa and Nigro (1980). Some of the current developments of hybrid methods which are implemented in constant step-size are contributed for example by Tsitouras (2002, 2003, 2006) and Franco (2006). Tsitouras has developed an explicit hybrid method of algebraic order seven with four stages per step intended for solving linear second-order problems of the form (1). He was also proposed two explicit hybrid methods of algebraic order six, one with six stages while the other with five stages per step, and has derived an implicit hybrid method of algebraic order eight with six stages per step. Meanwhile, Franco has derived explicit hybrid methods which reach up to order five and six with three and four stages per step, respectively. Other current research on hybrid methods include Coleman (2003) and Chan (2004) who have studied the hybrid methods theoretically through B-series and P-series, respectively. In this paper, based on explicit hybrid methods in Franco (2006), our attempt is to derive explicit hybrid methods of algebraic order seven with five stages per step. These methods will be compared with existing methods.

EXPLICIT HYBRID METHODS

We consider the class of explicit hybrid methods established by Franco (2006):

$$\left.\begin{aligned} & y = y_{n-1},\ Y_2 = y_n, \\ & Y_1 = (1+c)\,y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} f\left(x_n + c_j h, Y_j\right), i = 3,\ldots,s, \\ & y_{n+1} = 2y_n - y_{n-1} + h^2\left[b_1 f_{n-1} + b_2 f_n + \sum_{i=3}^{s} b_i f\left(x_n + c_i h, Y_i\right)\right]. \end{aligned}\right\} \tag{2}$$

where f_{n-1} and f_n represent $f(x_{n-1}, y_{n-1})$ and $f(x_n, y_n)$, respectively. These methods require $s-1$ function evaluations or stages per step and represented by the following table:

$$\begin{array}{c|cccccc} -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ c_3 & a_{31} & a_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array} \quad = \quad \begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}.$$

The order conditions for this class of methods are given by Coleman (2003). The leading term associated with the local truncation error of a p-th order explicit hybrid method is:

$$e_{p+1}(t_i) = \frac{\alpha(t_i)}{(p+2)!}\left[1+(-1)^{p+2} - \mathbf{b}^T \psi''(t_i)\right], t_i \in T_2, \rho(t_i) = p+2,$$

where $T_2,\rho(t_i)$, $\alpha(t_i)$ and $\Psi^n(t_i)$ are defined in Coleman (2003). The quantity:

$$E = \sqrt{\sum_{i=1}^{n_{p+2}} e_{p+1}^2(t_i)},$$

where n_{p+2} is the number of trees of order $p+2$, is called the error norm for the p-th order method.

BASIC THEORY

Let $H = \lambda h$ and $\mathbf{e} = (1 \ \ 1 \ \ \ldots \ \ 1)^T$. Applying the hybrid methods defined in (2) to equation, yields the recursion

$$y_{n+1} - S(H^2)y_n + P(H^2)y_{n-1} = 0 \tag{3}$$

where

$$S(H^2) = 2 - H^2\mathbf{b}^T(\mathbf{I} + H^2\mathbf{A})^{-1}(\mathbf{e} + \mathbf{c}),\ P(H^2) = 1 - H^2\mathbf{b}^T(\mathbf{I} + H^2\mathbf{A})^{-1}\mathbf{c}$$

The characteristic equation associated with (3) is:

$$\xi^2 - S(H^2)\xi + P(H^2) = 0 \tag{4}$$

According to Houwen and Sommeijer (1987), phase-lag is defined as the difference $t = H - \theta(H)$ where H is the phase (or argument) of the exact solution of $y'' = -\lambda^2 y$ and $\theta(H)$ is the phase of the principal root of (4). For the hybrid methods corresponding the characteristic equation (4), the quantity:

$$\phi(H) = H - \arccos\left(\frac{S(H^2)}{2\sqrt{P(H^2)}}\right),$$

is called phase-lag (or dispersion error) while the quantity

$$d(H) = 1 - \sqrt{P(H^2)}, \tag{5}$$

is called dissipation (or amplification) error. A hybrid method corresponding to (4) is said to have the phase-lag of order n if $\varphi(H) = O(H^{n+1})$. If $P(H^2) = 1$ then $d(H) = 0$ and the method having this property is said to be zero dissipative or dissipative of order infinity. If $P(H^2) \neq 1$ then $d(H) = O(H^{m+1})$ and the method with this property is said to be dissipative of order m. The interval $(0, H_p)$ is called the interval of periodicity of the method if:

$$P(H^2) = 1 \text{ and } |S(H^2)| < 2 \text{ for all } H \in (0, H_p),$$

where as the interval $(0, H_a)$ is called the interval of absolute stability if:

$$|P(H^2))| < 1 \text{ and } |S(H^2)| < 1 + P(H^2) \text{ for all } H \in (0, H_a).$$

DERIVATION OF THE NEW METHODS

We derive five-stage seventh order explicit hybrid methods algebraically. The new methods must satisfy the order conditions as given by Coleman (2003) with $s = 6$. There are 33 order conditions involved. By applying the simplifying condition:

$$\sum_{j=1}^{6} a_{ij}c_j^{\alpha} = \frac{c_i^{\alpha+2} + (-1)^{\alpha} c_i}{(\alpha+1)(\alpha+2)}, \alpha = 0,1, \tag{6}$$

some order conditions which are combinations of the other order conditions are eliminated leaving 15 order conditions. Substituting in $c_1 = -1$, $c_2 = 0$, $a_{21} = 0$ and $a_{ij} = 0$ for $j \geq i$ to the order conditions we get equations to be solved for the new methods. Other equations to be solved are obtained by substituting in $c_1 = -1$, $c_2 = 0$, $a_{21} = 0$ and $a_{ij} = 0$ for $j \geq i$ to the simplifying conditions (6). The following are all equations to be solved for the new methods.

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = 1. \tag{7}$$

$$-b_1 + b_3c_3 + b_4c_4 + b_5c_5 + b_6c_6 = 0 \tag{8}$$

$$b_1 + b_3c_3^2 + b_4c_4^2 + b_5c_5^2 + b_6c_6^2 = \tfrac{1}{6}. \tag{9}$$

$$-b_1 + b_3c_3^3 + b_4c_4^3 + b_5c_5^3 + b_6c_6^3 = 0. \tag{10}$$

$$b_1 + b_3c_3^4 + b_4c_4^4 + b_5c_5^4 + b_6c_6^4 = \tfrac{1}{15}. \tag{11}$$

$$b_3a_{31} + b_4a_{41} + b_5a_{51} + b_6a_{61} + b_4a_{43}c_3^2 + b_5a_{53}c_3^2 + b_6a_{63}c_3^2 + b_5a_{54}c_4^2 + b_6a_{64}c_4^2 + b_6a_{65}c_5^2 = \tfrac{1}{180}. \tag{12}$$

$$-b_1 + b_3c_3^5 + b_4c_4^5 + b_5c_5^5 + b_6c_6^5 = 0. \tag{13}$$

$$\begin{aligned}&b_3c_3a_{31}+b_4c_4a_{41}+b_5c_5a_{51}+b_6c_6a_{61}+b_4c_4a_{43}c_3^2\\&+b_5c_5a_{53}c_3^2+b_6c_6a_{63}c_3^2+b_5c_5a_{54}c_4^2+b_6c_6a_{64}c_4^2\\&+b_6c_6a_{65}c_5^2=\tfrac{1}{72}.\end{aligned} \tag{14}$$

$$\begin{aligned}&-b_3a_{31}-b_4a_{41}-b_5a_{51}-b_6a_{61}+b_4a_{43}c_3^3+b_5a_{53}c_3^3\\&+b_6a_{63}c_3^3+b_5a_{54}c_4^3+b_6a_{65}c_5^3=0.\end{aligned} \tag{15}$$

$$b_1+b_3c_3^6+b_4c_4^6+b_5c_5^6+b_6c_6^6=\tfrac{1}{28}. \tag{16}$$

$$\begin{aligned}&b_3c_3^2a_{31}+b_4c_4^2a_{41}+b_5c_5^2a_{51}+b_6c_6^2a_{61}+b_4c_4^2a_{43}c_3^2\\&+b_5c_5^2a_{53}a_3^2+b_6c_6^2a_{63}c_3^2+b_5c_5^2a_{54}c_4^2+b_6c_6^2a_{64}c_4^2\\&+b_6c_6^2a_{65}c_5^2=\tfrac{1}{336}.\end{aligned} \tag{17}$$

$$\begin{aligned}&-b_3c_3a_{31}-b_4c_4a_{41}-b_5c_5a_{51}-b_6c_6a_{61}\\&+b_4c_4a_{43}c_3^3+b_5c_5a_{53}c_3^3+b_6c_6a_{63}c_3^3\\&+b_5c_5a_{54}c_4^3+b_6c_6^2a_{65}c_5^2=-\tfrac{11}{1680}.\end{aligned} \tag{18}$$

$$\begin{aligned}&b_3a_{31}+b_4a_{41}+b_5a_{51}+b_6a_{61}\\&+b_4a_{43}c_3^4+b_5a_{53}c_3^4+b_6a_{63}c_3^4\\&+b_5a_{54}c_4^4+b_6a_{64}c_4^4+b_6a_{65}c_4^4\\&+b_6a_{65}c_5^4=\tfrac{1}{840}.\end{aligned} \tag{19}$$

$$\begin{aligned}&-b_4a_{43}c_3a_{31}-b_5a_{53}c_3a_{31}-b_6a_{63}c_3a_{31}-b_5a_{54}c_4a_{41}\\&-b_6a_{64}c_4a_{41}-b_6a_{65}c_5a_{51}+b_5a_{54}c_4a_{43}c_3+b_6a_{64}a_{43}c_3\\&+b_6a_{65}c_5a_{53}c_3+b_6a_{65}c_5a_{54}c_4=-\tfrac{11}{15120}.\end{aligned} \tag{20}$$

$$\begin{aligned}&b_4a_{43}a_{31}+b_5a_{53}a_{31}+b_6a_{63}a_{31}+b_5a_{54}a_{41}\\&+b_6a_{64}a_{41}+b_6a_{65}a_{51}+b_5a_{54}a_{43}c_3^2+b_6a_{64}a_{43}c_3^2\\&+b_6a_{65}a_{53}c_3^2+b_6a_{65}a_{54}c_4^2=\tfrac{1}{10080}.\end{aligned} \tag{21}$$

$$a_{31}+a_{32}=\frac{c_3^2+c_3}{2}. \tag{22}$$

$$a_{41}+a_{42}+a_{43}=\frac{c_4^2+c_4}{2}. \tag{23}$$

$$a_{51}+a_{52}+a_{53}+a_{54}=\frac{c_5^2+c_5}{2}. \tag{24}$$

$$a_{61}+a_{62}+a_{63}+a_{64}+a_{65}=\frac{c_6^2+c_6}{2}. \tag{25}$$

$$a_{31}=\frac{c_3-c_3^3}{6}. \tag{26}$$

$$-a_{41}+a_{43}c_3=\frac{c_4^3-c_4}{6}. \tag{27}$$

$$-a_{51}+a_{53}c_3+a_{54}c_4=\frac{c_5^3-c_5}{6}. \tag{28}$$

$$-a_{61}+a_{63}c_3+a_{64}c_4+a_{65}c_5=\frac{c_6^3-c_6}{6}. \tag{29}$$

From equation (26), we already have a_{31} in term of c_3. Solving equations (7) — (11) and (13) for b_i we get

$$b_1=\frac{5c_4c_3c_5+5c_6c_3c_4+2c_4+5c_6c_5c_4+2c_3+2c_6+5c_6c_5c_3 2c_5}{30(1+c_5)(c_4+1)(c_3+1)(1+c_6)}, \tag{30}$$

$$b_2=\frac{\begin{array}{c}30c_4c_3c_5c_6-5c_6c_5c_4-5c_6c_3c_4-2c_4+5c_4c_3\\-5c_4c_3c_5+5c_4c_5+5c_6c_4+5c_6c_3-2c_6+2\\-2c_3-5c_6c_5c_3-2c_5+5c_3c_5+5c_6c_5\end{array}}{30c_4c_3c_5c_6}, \tag{31}$$

$$b_3=-\frac{\begin{array}{c}5c_6c_5c_4-5c_6c_5-5c_6c_4+2c_6\\-5c_4c_5+2c_4-2c_4-2+2c_5\end{array}}{30(-c_6+c_3)(-c_4+c_3)(-c_5+c_3)c_3(c_3+1)}, \tag{32}$$

$$b_4=\frac{\begin{array}{c}-5c_6c_3-5c_3c_5+2c_3+5c_6c_5c_3\\-2+2c_5-5c_6c_5+2c_6\end{array}}{30c_4(-c_6+c_4)(-c_5+c_4)(c_4+1)(-c_4+c_3)}, \tag{33}$$

$$b_5=-\frac{\begin{array}{c}5c_6c_3c_4-5c_4c_3+2c_3-5c_6c_3\\+2c_4-2-5c_6c_4+2c_6\end{array}}{30(c_5-c_6)c_5(1+c_5)(c_4-c_5)(c_3-c_5)}, \tag{34}$$

$$b_6=\frac{\begin{array}{c}5c_4c_3c_5-5c_4c_5-5c_3c_5+2c_5\\-5c_4c_3+2c_4+2c_3-2\end{array}}{30(c_5-c_6)c_6(1+c_6)(-c_6+c_4)(-c_6+c_3)}. \tag{35}$$

Substituting (30) — (35) into equation (16) and solving the resulted equation for c_6 we get

$$c_6=\frac{-28c_4c_5-28c_3c_5+28c_4-15+70c_4c_3c_5+28c_5-28c_4c_3+28c_3}{14(5c_4c_3c_5-5c_4c_5-5c_3c_5+2c_5-5c_4c_3+2c_4+2c_3-2)}. \tag{36}$$

From equations (27) — (29), we have

$$a_{41}=a_{43}c_3-\left(\frac{c_4^3+c_4}{6}\right), \tag{37}$$

$$a_{51}=a_{53}c_3+a_{54}c_4-\left(\frac{c_5^3+c_5}{6}\right), \tag{38}$$

$$a_{61}=a_{63}c_3+a_{64}c_4+a_{65}c_5-\left(\frac{c_6^3+c_6}{6}\right). \tag{39}$$

Substituting (37) — (39) into equations (12), (14) and (17), then solving the resulted equations for a_{43}, we get

$$a_{43}=\frac{\left(28c_6c_5+14c_6+14c_5+15\right)c_4\left(c_4+1\right)\left(-c_4+c_3\right)}{168\left(c_3+1\right)c_3\left(-5c_6c_3+5c_6c_3-5c_3c_5+2c_3-2+2c_5-5c_6c_5+2c_6\right)}. \tag{40}$$

Substituting (37) — (39) into equations (12), (14), (15) and (18), then solving the resulted equations for a_{54} we get

$$a_{54}=-\frac{14c_3+28c_6c_3-28c_6+37}{5040\left(1+c_4\right)\left(-c_4+c_3\right)c_4\left(-c_6+c_5\right)b_5}. \tag{41}$$

Substitute (37) — (39) into equation (12), then the resulted equation is multiplied by c_6 and denoted equation A. Equation (14) is subtracted from equation A. Solving the resulted equation for a_{53} we get:

$$a_{53}=-\frac{\begin{array}{c}2c_6+1+360b_4a_{43}c_4c_3+360b_4a_{43}c_4c_3^2-360b_4a_{43}c_6c_3-360b_4a_{43}c_6c_3^2\\ -360b_5a_{54}c_6c_4-360b_5a_{54}c_6c_4^2+360b_5a_{54}c_5c_4+360b_5a_{54}c_5c_4^2\end{array}}{360b_5c_3\left(-c_6+c_5\right)\left(1+c_3\right)}. \tag{42}$$

Substituting (37) — (42) into equations (15) and (20), then solving the resulted equations for a_{64} and a_{65} we get:

$$a_{64}=\frac{-28c_5^2c_3+14c_5c_3+14c_4c_3-28c_6c_3+28c_6c_3c_4+28c_5^2-43c_5+6c_6+37c_4-28c_6c_4}{5040b_6c_4(1+c_4)(-c_4+c_3)(-c_6+c_5)(-c_5+c_4)}, \tag{43}$$

$$a_{65}=-\frac{14c_4c_3-14c_3+3-14c_4}{2520b_6c_5\left(c_4c_3+c_5c_4c_3-c_5c_3-c_5^2c_3-c_5c_4-c_5^2c_4+c_5^3+c_5^2\right)}. \tag{44}$$

Substituting (37) — (44) into equation (19) and solving the resulted equation for a_{63} we get:

$$a_{63}=\frac{\begin{array}{l}-37c_3-6c_6+43c_5+28c_6c_3-5040b_4a_{43}c_3^2c_4^2\\ +5040b_4a_{43}c_3^3c_4+28c_6c_4-14c_4c_3-28c_6c_3c_4+5040b_4a_{43}c_3^2c_5^2\\ -5040b_4a_{43}c_3^3c_5+5040b_4a_{43}c_3^3c_5^2+5040c_4^2b_4a_{43}c_3^2c_5\\ -28c_5^2+5040b_4a_{43}c_3^4c_4-5040b_4a_{43}c_3^3c_4^2-14c_5c_4+28c_5^2c_4\\ -5040c_4b_4a_{43}c_3^2c_5^2+5040c_4^2b_4a_{43}c_3c_5\\ -5040c_4b_4a_{43}c_3c_5^2-5040b_4a_{43}c_3^4c_5\end{array}}{5040b_6c_3(c_3-c_5)(-c_4+c_3)(-c_6+c_5)(1+c_3)}. \tag{45}$$

Substituting (33) — (35), (37) — (44) into equation (21) and solving the resulted equation for c_3 we get $-1\frac{1}{2}+\frac{1}{2}\sqrt{5}$. Substituting (33) — (44) and c_3 into equation (17) and solving the resulted equation for c_4 we get $c_4 = 1$. Lastly, from equations (22), (23) and (24) we get:

$$a_{32}=\frac{c_3^2+c_3}{2}-a_{31},\ a_{42}=-a_{41}-a_{43},\ a_{52}=-a_{51}-a_{53}-a_{54},$$

$$a_{62}=\frac{c_6^2+c_6}{2}-a_{61}-a_{63}-a_{64}-a_{65}.$$

Now the only free parameter left is c_5. We obtain two methods depending on the value of c_5. As for the error norm, E of our methods, the computation of E has been done using trees related with the sum in the order conditions (46) to (53) listed in Table 1 (Coleman 2003).

TABLE 1. Order conditions associated with trees of order 9

Order conditions	
$\sum_{i=1}^{s} b_i c_i^7 = 0$	(46)
$\sum_{i=1}^{s}\sum_{j=1}^{s} b_i c_i^3 a_{ij} c_j^2 = \frac{1}{180}$	(47)
$\sum_{i=1}^{s}\sum_{j=1}^{s} b_i c_i^2 a_{ij} c_j^3 = 0$	(48)
$\sum_{i=1}^{s}\sum_{j=1}^{s} b_i c_i a_{ij} c_j^4 = \frac{1}{180}$	(49)
$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i c_i a_{ij} a_{jk} c_k^2 = \frac{1}{1080}$	(50)
$\sum_{i=1}^{s}\sum_{j=1}^{s} b_i a_{ij} c_j^5 = 0$	(51)
$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i a_{ij} c_j a_{jk} c_k^2 = \frac{1}{2160}$	(52)
$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i a_{ij} a_{jk} c_k^3 = 0$	(53)

Strategies employed in choosing the free parameter for our methods are:

1. Minimize the function P which is given by $P=\sqrt{\sum_{i=1}^{8} s_i^2}$.

   Here, s_i's represent expressions obtained by subtracting the right sides of order conditions (46) to (53) from the left sides.
2. Increase the dissipation order.

For our first method, we select the parameter c_5 so that P is as small as possible obtaining the values $c_5=\frac{28521}{50000}$, $E = 6.755534178017401 \times 10^{-4}$. This method has an interval of absolute stability (0, 3.341) while the phase-lag and dissipation error are respectively given by:

$$\varphi(H)=\left(-\frac{30585671}{3670272000000}+\frac{555412997}{128459520000000}\sqrt{5}\right)$$

$H^9+O(H^{11})$ and

$d(H) = 1.921345174 \times 10^{-5} H^8 + O(H^{10})$

Thus, this method which will be denoted as EHM7(8, 7) has the phase-lag of order 8 and dissipative of order 7.

For our second method, c_5 is selected so that the dissipation order is increased. To do this, we first establish conditions related with dissipation error that have to be satisfied by the five-stage explicit hybrid methods. The expressions $S(H^2)$ and $P(H^2)$ for the five-stage explicit hybrid methods are polynomials in H^2 given by:

$$S(H^2) = 2 - u_1H^2 + u_2H^4 - u_2H^6 + u_3H^8 - u_5H^{10}.$$

$$P(H^2) = 1 - t_1H^2 + t_2H^4 - t_3H^6 + t_4H^8 - t_5H^{10}.$$

where u_i and t_i are expressions in terms of coefficients of the method. Expanding equation (5) in Taylor series, we obtain:

$$d(H) = \frac{1}{2}t_1H^2 + \left(-\frac{1}{2}t_2 + \frac{1}{8}t_1^2\right)H^4 + \left(\frac{1}{2}t_3 - \frac{1}{4}t_1t_2 + \frac{1}{16}t_1^3\right)$$
$$H^6 + \left(-\frac{1}{2}t_4 + \frac{1}{4}t_1t_3 + \frac{1}{8}t_2^2 - \frac{3}{16}t_1^2t_2 + \frac{5}{128}t_1^4\right)H^8 + O\left(H^{10}\right)$$

By setting the coefficients of H^{2i}, $i \geq 1$ to zero, we obtain conditions for the explicit hybrid methods to have the dissipation of order 9. We note that this is the highest attainable dissipation order for our method.

Dissipation of order 9: Conditions to be satisfied are

$$\frac{1}{2}t_1 = 0, -\frac{1}{2}t_2 + \frac{1}{8}t_1^2 = 0, \frac{1}{2}t_3 - \frac{1}{4}t_1t_2 + \frac{1}{16}t_1^3 = 0$$

and

$$-\frac{1}{2}t_4 + \frac{1}{4}t_1t_3 + \frac{1}{8}t_2^2 - \frac{3}{16}t_1^2t_2 + \frac{5}{128}t_1^4 = 0.$$

Substituting (30) — (45), and $c_4 = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$ into the equations corresponding to the above conditions and solving the resulted equations for c_5 yields $c_5 = \frac{406}{16347} - \frac{50993}{179817}\sqrt{5}$ and the error norm is $E = 6.156459599162350 \times 10^{-3}$. This method has an interval of absolute stability (0, 2.843]. The phase-lag and dissipation error for this method are given by:

$$\varphi(H) = -\frac{13}{7257600}H^9 + O\left(H^{11}\right)$$

and $d(H) = -2631856949 \times 10^{-7} H^{10} + O(H^{20})$.

Therefore, this method which will be denoted as EHM7(8, 9) has the phase-lag of order 8 and dissipative of order 9. Most of the values of coefficients of EHM7(8, 7) and EHM7(8, 9) methods are in a form of surds and are too long to be reported in this paper. A complete list of these values will be given to the reader upon request.

PROBLEMS TESTED

The following are some second-order problems with exact solutions taken from the literature that have been used to evaluate the performance of our methods.

Problem 1 In-homogenous problem
$y'' = -100y + 99\sin(x)$, $y(0) = 1, y'(0) = 11$, $x \in [0,5]$
Solution: $y(x) = \cos(10x) + \sin(10x) + \sin(x)$.

Problem 2 Homogeneous problem
$y'' = -y$, $y(0) = 0$, $y'(0) = 1$, $x \in [0,10]$
Solution: $y(x) = \sin(x)$.

Problem 3 Nonlinear oscillatory problem

$$y''_1 = -4x^2y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}}, \; y_1(0),\; y'_1(0) = 0$$

$$y''_2 = -4x^2y_2 + \frac{2y_2}{\sqrt{y_1^2 + y_2^2}}, \; y_2(0) = 0,\; y'_2(0) = 0,\; x \in [0,5]$$

Solution: $y_1(x) = \cos(x^2)$, $y_2 = \sin(x^2)$.

Problem 4 Orbit problem

$$y''_1 = -\frac{y_1}{\left(\sqrt{y_1^2 + y_2^2}\right)^3}, \; y_1(0) = 1 - e,\; y'_1(0) = 0$$

$$y''_2 = -\frac{y_1}{\left(\sqrt{y_1^2 + y_2^2}\right)^3}, \; y_2(0) = 0,\; y'_2(0) = \sqrt{\frac{\sqrt{1+e}}{1-e}}, \; x \in [0,10]$$

with e representing the eccentricity of the orbit. The theoretical solution of this problem is $y_1(x) = \cos(R) - e$, $y_2(x) = \sqrt{1-e^2}\sin(R)$, where R satisfies the Kepler's equation $x = R - e\sin(R)$. In this paper, we only consider the case $e = 0$.

Problem 5 Almost orbit problem
$z''(x) + z(x) = e^{ix}$, $z(0) = 1$, $z'(0) = 0.9995i$, $z \in \mathbf{C}$, $x \in [0,20]$

The theoretical solution is $z(x) = (1 - 0.0005ix)e^{ix}$. In this paper, we solve the equivalent system

$$y''_1 = y_1 + \frac{1}{1000}\cos(x),\; y_1(0) = 1,\; y'_1(0) = 0$$

$$y''_2 = y_2 + \frac{1}{1000}\sin(x),\; y_2(0) = 0,\; y'_2(0) = 0.9995$$

with the theoretical solution: $y_1(x) = \cos(x) + 0.0005x\sin(x)$, $y_2(x) = \sin(x) - 0.0005x\cos(x)$.

Problem 6 Periodically forced nonlinear problem (undamped Duffing's equation)
$y'' = -y - y^3 + (\cos(x) + \varepsilon\sin(10x))^3 - 99\varepsilon\sin(10x)$,
$y(0) = 1$, $y'(0) = 10\varepsilon$, $x \in [0.20]$.
We choose $\varepsilon = 10^{-3}$. Solution: $y(x) = \cos(x) + \varepsilon\sin(10x)$.

NUMERICAL RESULTS

The methods that have been used in the comparisons are denoted by:

1. TSI6: The sixth-order explicit hybrid method with five stages found in Tsitouras (2003)
2. TSI7: The seventh-order explicit hybrid method with four stages derived in Tsitouras (2002)
3. RKNH2: The second-order explicit RKN method with five stages derived by van Der Houwen and Sommeijer (1987)

4. EHM7(8, 7): The first seventh-order explicit hybrid method with five stages derived in this paper.
5. EHM7(8, 9): The second seventh-order explicit hybrid method with five stages derived in this paper.

The numerical results shown in Figures 1, 3 and 4 have been computed with step-size $h = 0.1/2^i$, $i = 0,1,\ldots,4$ for EHM7(8, 7), EHM7(8, 9), TSI7 and RKNH2 methods, and with step-size $h = 0.01/2^i$, $i = 2,3,\ldots,6$ for TSI6. In Figure 2, the numerical results for EHM7(8, 7), EHM7(8, 9), TSI7 and RKNH2 methods have been computed with $h = 1/2^i$, $i = 0,1,\ldots,4$ while for TSI6, $h = 0.1/2^i$, $i = 0,1,\ldots,4$. Figure 5 presents the numerical results for EHM7(8, 7), EHM7(8, 9), TSI7 and RKNH2 using $h = 1/2^i$, $i = 0,1,\ldots,4$ while for TSI6 method, $h = 0.1/2^i$, $i = 4,5,\ldots,8$. In Figure 6, the step-sizes used in the computation of the numerical results for EHM7(8, 7), TSI7, EHM7(8, 9) and RKNH2 methods are $h = 0.1/2^i$, $i = 0,1,\ldots,4$ while for TSI6 method, $h = 0.1/2^i$, $i = 1,2,\ldots,5$. In Figure 1 and Figure 2, curves of $\log_{10}$(end-point error) versus step-size are depicted. The formula of the maximum global error (MAXGE) is $\text{MAXGE} = \|y(x_n) - y_n\|$ where $y(x_n)$ is the exact solution and y_n is the numerical solution. Figure 3 – 6 display the curves of $\log_{10}$(MAXGE) versus step-size whereas Figure 7 shows the total time (in seconds) required by each method to solve each problem over various step-sizes. The horizontal grid tick-marks in Figure 7 represent the problems solved where:

1. 0 means "Problem 1",
2. 1 means "Problem 2",
3. 2 means "Problem 3",
4. 3 means "Problem 4",
5. 4 means "Problem 5", and
6. 5 means "Problem 6".

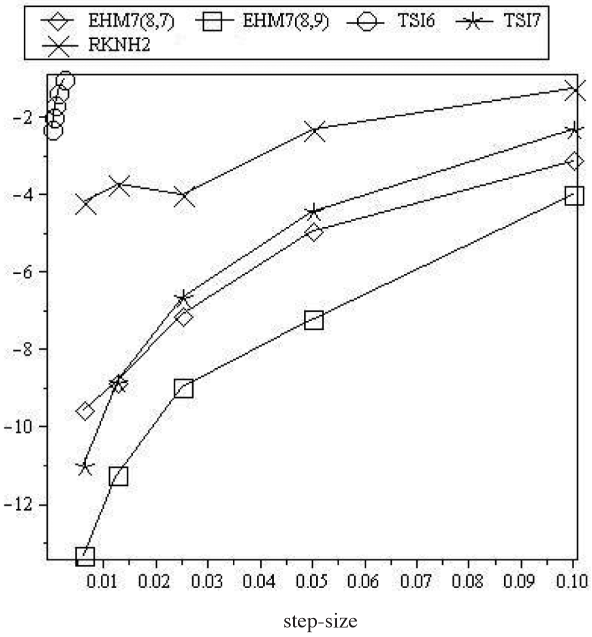

FIGURE 1. $\log_{10}$(end-point error) versus step-size for Problem 1

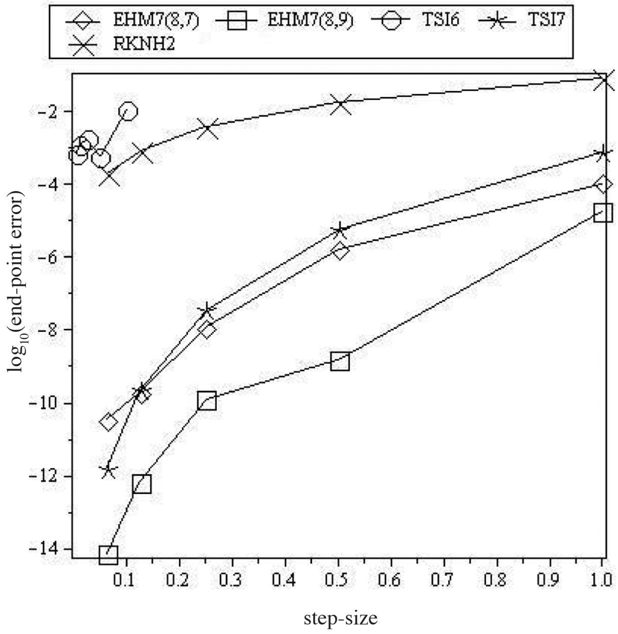

FIGURE 2. $\log_{10}$(end-point error) versus step-size for Problem 2

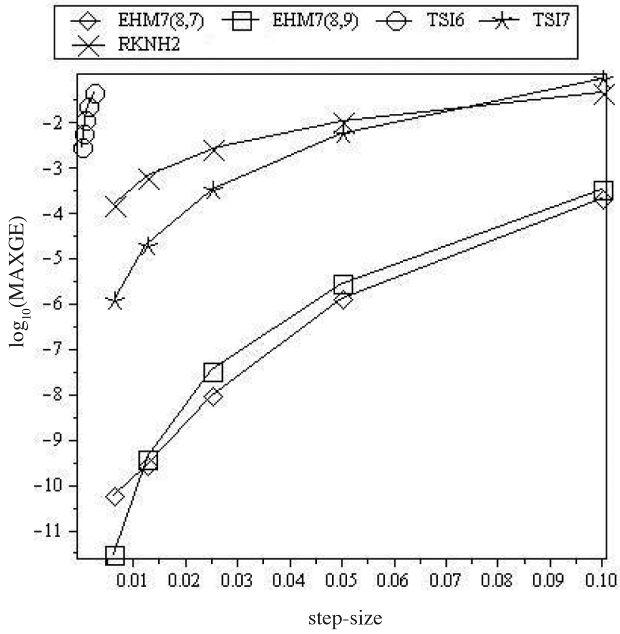

FIGURE 3. $\log_{10}$(MAXGE) versus step-size for Problem 3

DISCUSSION AND CONCLUSION

From the numerical results in Figures 1 to 6, we observe that EHM7(8, 9) method is the most efficient for solving Problems 1, 2, 4, and 6 of all methods being compared. On the other hand, TSI6 is the least efficient method. For Problem 6, EHM7(8, 7) and EHM7(8, 9) both perform well whereas for Problem 3, EHM7(8, 7) method gives the best performance. From Figure 7, it is obvious that the total time required by TSI6 to solve each problem over various step-sizes is longer than that required by other methods

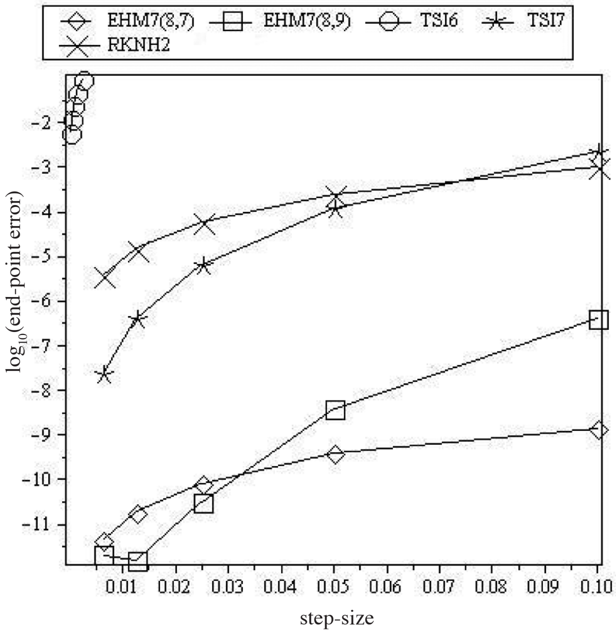

FIGURE 4. Log_{10}(MAXGE) versus step-size for Problem 4

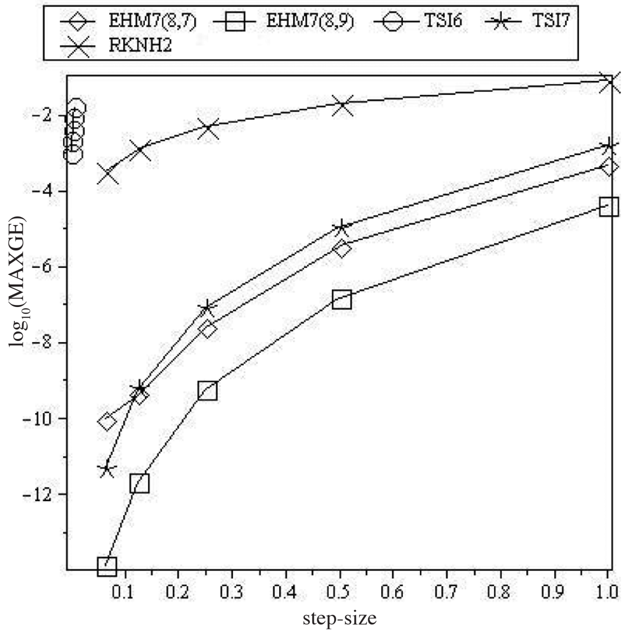

FIGURE 5. Log_{10}(MAXGE) versus step-size for Problem 5

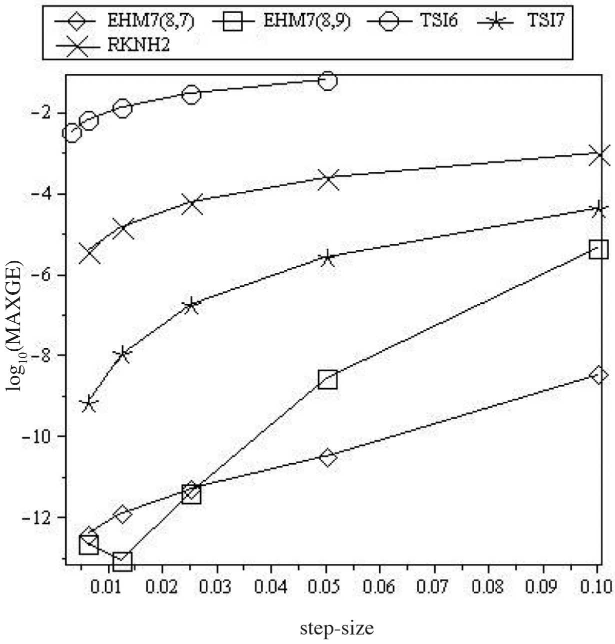

FIGURE 6. Log_{10}(MAXGE) versus step-size for Problem 6

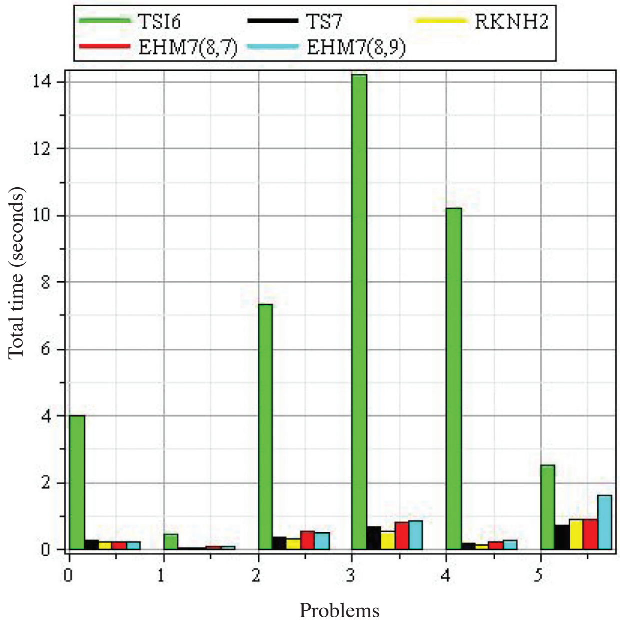

FIGURE 7. Total time required by each method to solve each problem

considered. It is because the computation of TSI6 code needs smaller step-sizes which results in more number of function evaluations in order to gain accuracy. It is also clear that for Problems 3, 4 and 5, the computations of RKNH2 and TSI7 codes are faster than that of EHM7(8, 7) and EHM7(8, 9) codes whereas for Problem 6, TSI7 is the fastest of all codes considered. In conclusion, the new methods perform more efficiently than TSI6, TSI7 and RKNH2 methods. Furthermore, the new method with reduced dissipation error is preferable for solving most of the problems used in this paper. All codes are designed using Microsoft Visual C++ version 6.0 in HP computer with Intel(R)Core(TM)2Duo CPU P8600@2.40GHz.

REFERENCES

Brusa, L. & Nigro, L. 1980. A one-step method for direct integration of structural dynamic equations. *International Journal for Numerical Methods in Engineering* 15: 685-699.

Cash, J.R. 1981. High order P-stable formulae for the numerical integration of periodic initial value problems. *Numerische Mathematik* 37: 355-370.

Chan, R.P.K., Leone, P. & Tsai, A. 2004. Order conditions and symmetry for two-step hybrid methods. *International Journal of Computer Mathematics* 81(12): 1519-1536

Chawla, M.M. 1984. Numerov made explicit has better stability. *BIT* 24: 117-118.

Coleman, J.P. 2003. Order conditions for a class of two-step methods for $y'' = f(x,y)$. *Journal of Numerical Analysis* 23: 197-220.

Fatunla, S.O., Ikhile M.N.O. & Otunta, F.O. 1999. A class of P-stable linear multistep numerical methods. *International Journal of Computer Mathematics* 72: 1-13.

Franco, J. M. 2006. A class of explicit two-step hybrid methods for second-order IVPs. *Journal of Computational .and Applied Mathematics* 187: 41-57.

Hairer, E. 1979. Unconditionally stable methods for second order differential equations, *Numerische Mathematik* 32: 373-379.

Tsitouras, Ch. 2002. Explicit two-step methods for second-order linear IVPs. *Computers and Mathematics with Applications* 43: 943-949.

Tsitouras, Ch. 2003. Families of explicit two-step methods for integration of problems with oscillating solutions. *Applied Mathematics and Computation* 135: 169-178.

Tsitouras, Ch. 2006. Stage reduction on P-stable Numerov type methods of eighth order. *Journal of Computational and Applied*. *Mathematics* 191: 297-305.

van Der Houwen, P.J & Sommeijer, B.P. 1987. Explicit Runge-Kutta (-Nystrom) methods with reduced phase errors for computing oscillating solutions. *SIAM Journal of Numerical Analysis* 24: 595-617.

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